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# A Cauchy integral-equation method for the numerical solution of the steady water-wave problem

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SUMMARY

In this paper a method is developed for the numerical solution of singular integral equations related to the water-wave problem. Periodic gravity waves of constant form in water of finite depth have been studied. The problem has been programmed and run on a computer, and the computed results plotted and compared with those of other authors. Some difficulties of the computing and some checking of the solution are discussed.

## 1. Introduction

The problem of free streamlines is an old and difficult one in hydrodynamics. The difficulty arises from the fact that it is a mixed Dirichlet-Neumann boundary-value problem, in which the positions of part of the boundary, the free streamlines, are unknown, and along the free streamline, the pressure is constant.

The problem of a steady progressive wave is such a free-surface problem. The theory of water waves was first investigated by Gerstner [4] in 1804. In 1847, Stokes [16] investigated the motion of a water wave of constant form and finite amplitude to third order of approximation, for water of infinite depth, and to second order of approximation when the depth is finite. Later, in 1880 [17] he reworked the problem, using the velocity potential and stream function, rather than the space coordinates, as independent variables. He obtained the fifth order of approximation for infinite depth and the third order of approximation for finite depth of water. De [2] extended the works of Stokes to fifth order of approximation. Recently (1974) Schwartz [14] extended the approximation to very high order (30 to 115 terms) using a modern digital computer to perform the coefficient arithmetic, and used Padé approximation to sum the series and continue it analytically.

A proof of the existence of periodic gravity waves of constant form in water of infinite depth was first given by Nekrasov [13] and later independently by Levi-Civita [10]. Nekrasov formulated the problem as a nonlinear integral equation, and showed that a non-trivial solution could be found for sufficiently small values of wave amplitudes. Levi-Civita's formulation of the problem is essentially the same as Nekrasov's. He established the existence of the solution by establishing the convergence of a series in amplitude-to-wavelength $\mu$, for sufficiently small values of $\mu$. No estimate of a radius of convergence was given. Struik [18] extended the work of Levi-Civita to water of finite depth. Some errors of Struik's paper have been corrected by Hunt [6]. Krasovskii [8] in 1960 gave a proof using the theory of positive operators. He showed that Nekrasov's integral equation has a solution for any depth, provided that the maximum surface angle does not exceed $\pi/6$.

The existence of highest progressive waves is still an open problem. In 1880, Stokes [17], assuming that such a wave existed, showed that the maximum surface angle should be $\pi/6$. Michell [12], Havelock [5], Nekrasov [13] and Yamada [23] also investigated the highest progressive waves and Michell [12] showed, using series, that the amplitude-to-wavelength ratio is approximately 1 : 7.

In this paper, we consider the water-wave problem, using a numerical method based on integral equations. In 1925, Lauck [9] investigated the form of the free streamline for the flow over a weir of infinite depth, and at rest at infinity, using Cauchy's integral formula. He made an assumption that the flow at infinity should be radial in the $z$-plane and solved the integral equations successively for $x$ and $y$ by graphical integration. In 1963, Jaswon [7], Symm [19] and others used Fredholm's first integral equation and Green's boundary formula to solve numerically some potential problems particularly in elasticity. The numerical technique used will be discussed briefly in later sections. Jaswon [7] mentioned that the integral-equation method for solving the boundary-value problems of potential theory and classical elasticity has not been greatly exploited. He also indicated three reasons for this: firstly, most integral equations of physical significance involve singular or weakly singular kernels; secondly, the integral-equation method will eventually involve solving a fairly large number of simultaneous linear algebraic equations which can only be handled by modern digital computers; lastly, a given integral equation may, or may not, have a solution. Since high-speed computers have been developed, the second difficulty will no longer be so important. The first difficulty in most cases could be overcome by developing some higher-order formulas for the equations around the singularities.

The problem will be formulated in Section 2. In Section 3, we derive the integral equations for solving the problem, and discuss the connection of our equation with Jaswon's [7]. In Section 4, we discuss the numerical method we propose to use for solving the integral equation; in particular, we develop some approximate formulae for points near corners and around singularities. In Section 5, we compare the computed results with those of Thomas [20], Conway and Thomas [1], Schwartz [14] and Wehausen and Laitone [22].

## 2. Formulation of the problem

Consider a symmetrical two-dimensional periodic wave moving from right to left with constant velocity $c$ on the surface of fluid of finite depth. If we superpose a constant velocity $c$ on the fluid from left to right, the motion becomes steady, and the motion of the fluid is from left to right. The fluid is assumed to be inviscid and incompressible, and the motion irrotational. The bottom of the fluid is assumed to be horizontal, and the depth from the undisturbed water level (the mean depth of the fluid) is $h$. We shall investigate the motion of the fluid contained in a half-wavelength, that is, the region $ABCD$ (see Fig. 1). Let the origin of the space coordinates $x$, $y$ be in a trough (that is, the point $D$ in Fig. 1); the $x$-axis from left to right; and the $y$-axis upward. The wavelength is denoted by $\lambda$ and the amplitude of the wave by $a$. Let the least depth of the fluid be $h_1$ and the mean elevation of the wave from the trough be $d$; then $h = h_1 + d$. The wave speed $c$ may be defined as

$$c = \frac{2}{\lambda} \int_0^{\frac{1}{2}\lambda} u(x, y) dx. \tag{1}$$

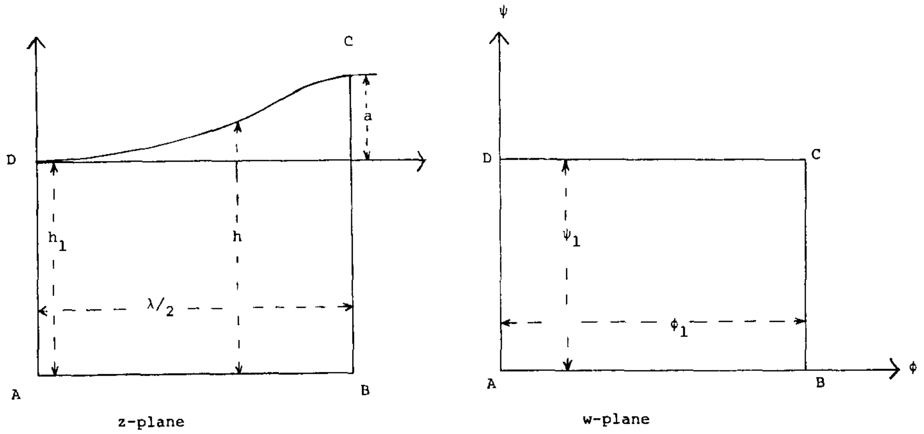


Figure 1.

This is the Stokes' second definition of the wave speed $c$. For more details of discussion of this, the reader should consult Stokes [16] or Wehausen and Laitone [22, pp. 456–7].

Let $\phi$ be the velocity potential and $\psi$ the stream function, and $w = \phi + i\psi$, $z = x + iy$. Then $dw/dz = u - iv = qe^{-i\theta}$. Now we map the fluid region $ABCD$ in the $z$-plane into $ABCD$ in the $w$-plane. Consider Bernoulli's equation along the free streamline $DC$, with density $\rho = 1$,

$$q^2 + 2gy = \text{constant} = q_0^2 \tag{2}$$

which may be written in integral form by differentiating with respect to $\phi$, replacing $dy/d\phi$ by $v/q^2 = (\sin\theta)/q$, and integrating to obtain

$$q^3 + 3g\int_0^{\phi} \sin\theta(t)dt = q_0^3, \tag{3}$$

where $q_0$ is the speed at $D$.

We now introduce dimensionless variables,

$$\bar{\phi} = \phi/\psi_1, \quad \bar{\psi} = \psi/\psi_1, \quad \bar{q} = q/q_0, \quad \bar{g} = g\psi_1/q_0^3$$

where $\psi_1$ is the volume flux across $AD$ (see Figure 1). Then (3) becomes

$$\bar{q}^3 + 3\bar{g}\int_0^{\bar{\phi}} \sin\theta(t)dt = \bar{q}_0^3 = 1.$$

We now drop all the bars, writing

$$q^3 + 3g\int_0^{\phi} \sin\theta(t)dt = q_0^3 = 1, \tag{4}$$

remembering that all variables hereafter are *dimensionless*. Then $q_0 = 1$ and $\psi_1 = 1$. Equation (4) is the free-surface condition in integral form.

If all $q$'s along $AB$, $BC$, $CD$ and $DA$, and $\theta$ along $CD$ have been computed, then the form of the free surface $(x, y)$, the wave speed $c$, the wavelength $\lambda$, the amplitude $a$, the mean elevation $d$, the height $h_1$, the mean depth $h$ and other quantities can be calculated. Here $\phi_1$ and $q_1$, the velocity at the crest $C$, are assumed to be given. Clearly from (2), all $y$'s are known. The $x$'s and $y$'s can also be computed from $x_\phi = (\cos\theta)/q$ and $y_\phi = (\sin\theta)/q$, that is

$$\begin{aligned} x(\phi) &= \int_0^\phi \frac{\cos\theta(t)}{q(t)}\,dt \text{ along } DC, \\ y(\phi) &= \int_0^\phi \frac{\sin\theta(t)}{q(t)}\,dt \text{ along } DC. \end{aligned} \tag{5}$$

Thus,

$$\lambda = 2x_c, \quad a = y_c, \quad \text{since } x_D = y_D = 0. \tag{6}$$

For the wave speed $c$,

$$c = \frac{2}{\lambda}\int_0^{\lambda/2} u(x, y)dx = \frac{2\phi_1}{\lambda}. \tag{7}$$

Let $T_1$ be the time required for a fluid particle to travel from $D$ to $C$ along the free surface $DC$. The mass-transport velocity $u_0$ along $DC$ is defined by

$$u_0 = c - \frac{\lambda}{2T_1}, \tag{8}$$

see Ursell [21] or Longuet-Higgins [11]. Moreover $T_1$ can be expressed in the form

$$T_1 = \int_0^{T_1} dt = \int_0^{\phi_1} \frac{1}{q(\phi)^2}\,d\phi \text{ along } DC. \tag{9}$$

The mean elevation $d$ of the wave above the trough is defined by

$$d = \frac{2}{\lambda}\int_D^C y(x)dx.$$

Levi-Civita [10] has shown that

$$c_1^2 + 2gd = q_0^2 \tag{10}$$

where

$$c_1^2 = \frac{2}{\lambda}\int_A^B u^2(x, y)dx = \frac{2}{\lambda}\int_0^{\phi_1} q(\phi)d\phi \text{ along } AB. \tag{11}$$

The depth $h_1$ of the fluid at the trough $D$ is

$$h_1 = \int_{-h_1}^0 dy = \int_0^1 \frac{1}{q(\psi)}\,d\psi \text{ along } AD. \tag{12}$$

## 3. Cauchy integral equations

Let $\Gamma$ be a simple closed contour, taken in the positive sense (counter-clockwise), such that the function $f(z)$ is analytic at every point of $\Gamma$ and its interior. Then the Cauchy integral formula is

$$2\pi i f(z_0) = \int_\Gamma \frac{f(z)}{z - z_0} dz \tag{13}$$

if $z_0$ is an interior point.

The left-hand side of (13) is zero if $z_0$ is an exterior point. Now, suppose $z_0$ is a point on the contour $\Gamma$, and $\Gamma$ is smooth at $z_0$. Then the Cauchy integral formula becomes

$$\pi i f(z_0) = \int_\Gamma \frac{f(z)}{z - z_0} dz. \tag{14}$$

However, when the slope of $\Gamma$ is discontinuous at $z_0$, with interior angle $\beta$, the Cauchy integral formula assumes the form

$$\beta i f(z_0) = \int_\Gamma \frac{f(z)}{z - z_0} dz.$$

In particular, when $\beta = \pi/2$, we obtain the formula at a corner

$$\frac{\pi}{2} i f(z_0) = \int_\Gamma \frac{f(z)}{z - z_0} dz. \tag{15}$$

Let $f(z) = U(x, y) + iV(x, y)$ and $z - z_0 = \rho e^{i\alpha}$. Let $s$ be the arc length along the contour $\Gamma$ and $\boldsymbol{n}$ the *inward* unit normal to $\Gamma$. Then the Cauchy-Riemann conditions are

$$\frac{\partial U}{\partial s} = \frac{\partial V}{\partial n}, \quad \frac{\partial U}{\partial n} = -\frac{\partial V}{\partial s},$$

$$\frac{\partial(\ln \rho)}{\partial s} = \frac{\partial \alpha}{\partial n}, \quad \frac{\partial(\ln \rho)}{\partial n} = -\frac{\partial \alpha}{\partial s}. \tag{16}$$

Now $z - z_0 = \rho e^{i\alpha}$ and hence $dz = (z - z_0)(d(\ln \rho) + i d\alpha)$ and equation (14) becomes

$$\pi[iU(x_0, y_0) - V(x_0, y_0)] = \int_\Gamma [U(x, y) + iV(x, y)][d(\ln \rho) + i d\alpha].$$

Equating the real and imaginary parts, we obtain

$$\pi U(x_0, y_0) = \int_\Gamma V(x, y) d(\ln \rho) + \int_\Gamma U(x, y) d\alpha,$$

$$\pi V(x_0, y_0) = -\int_\Gamma U(x, y) d(\ln \rho) + \int_\Gamma V(x, y) d\alpha. \tag{17}$$

These integral equations are those used by Lauck [9] to obtain the flow over a weir by means of graphical integration. Integrating by parts the first integrals of both equations of (17) and using (16), we obtain

$$\pi U(x_0, y_0) = \int_\Gamma \frac{\partial U(x, y)}{\partial n} \ln \rho \, ds - \int_\Gamma U(x, y) \frac{\partial(\ln \rho)}{\partial n} ds,$$

$$\pi V(x_0, y_0) = \int_\Gamma \frac{\partial V(x, y)}{\partial n} \ln \rho \, ds - \int_\Gamma V(x, y) \frac{\partial(\ln \rho)}{\partial n} ds. \tag{18}$$

Equations (18) are of the same form as the equation of Jaswon [7] (equation (18) of [7]) and also Green's boundary formula. Applying (16) again, we find that equations (17) and (18) can be rewritten in the following form:

$$\pi U(x_0, y_0) = -\int_\Gamma \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma U(x, y) \frac{\partial \alpha}{\partial s} ds,$$

$$\pi V(x_0, y_0) = \int_\Gamma \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma V(x, y) \frac{\partial \alpha}{\partial s} ds. \tag{19}$$

For a corner point $z_0$, (15) will become

$$\frac{\pi}{2} U(x_0, y_0) = -\int_\Gamma \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma U(x, y) \frac{\partial \alpha}{\partial s} ds,$$

$$\frac{\pi}{2} V(x_0, y_0) = \int_\Gamma \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \int_\Gamma V(x, y) \frac{\partial \alpha}{\partial s} ds. \tag{20}$$

In the next section, we obtain a method for solving the water-wave problem using (19) and (20) with $U = U(\phi, \psi) = \ln q(\phi, \psi)$ and $V = V(\phi, \psi) = -\theta(\phi, \psi)$.

## 4. The numerical scheme

Let $\Gamma$ be the rectangle $ABCD$ in the $w$-plane in Fig. 1, and $\phi_1$ and $q_1$ be known quantities. Divide each side of the rectangle into a number of equal subintervals. Then (19) and (20) can be written as the following discrete qums:

$$\pi U(x_0, y_0) = -\sum_j \int_j \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j U(x, y) \frac{\partial \alpha}{\partial s} ds,$$

$$\pi V(x_0, y_0) = \sum_j \int_j \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j V(x, y) \frac{\partial \alpha}{\partial s} ds, \tag{21}$$

when $(x_0, y_0)$ is not a corner point, and

$$\frac{\pi}{2} U(x_0, y_0) = -\sum_j \int_j \frac{\partial V(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j U(x, y) \frac{\partial \alpha}{\partial s} ds,$$

$$\frac{\pi}{2} V(x_0, y_0) = \sum_j \int_j \frac{\partial U(x, y)}{\partial s} \ln \rho \, ds + \sum_j \int_j V(x, y) \frac{\partial \alpha}{\partial s} ds, \tag{22}$$

for a corner point $(x_0, y_0)$, where the integrals in (21) and (22) range over all subintervals of $\Gamma$. We approximate each integral by Simpson's formula, that is,

$$\int_j f(x)dx = \int_{-h}^{h} f(x)dx = \frac{h}{3}[f(-h) + 4f(0) + f(h)] + O(h^5),$$

where $f(x)$ is any integrand of (21) and (22).

Symm [19] assumed that $U$, $V$, $\partial U/\partial s$, $\partial V/\partial s$ are constant in each subinterval and approximated the integrals with the remaining integrand. Hence the results Symm [19] obtained are exact only when $U$ and $V$ are constant; that is, it is a first-order approximation.

Let $h_x$ and $h_y$ be the lengths of the subintervals along the horizontal and vertical sides respectively. For each particular node, $w_0 = \phi_0 + i\psi_0$, one of the end points of the subintervals, we have to compute the values of $\ln \rho$ and $\partial\alpha/\partial s$, where $\rho^2 = (\phi - \phi_0)^2 + (\psi - \psi_0)^2$ and $\tan \alpha = (\psi - \psi_0)/(\phi - \phi_0)$ with $\phi + i\psi \neq \phi_0 + i\psi_0$. Note that if $\phi + i\psi \neq \phi_0 + i\psi_0$, then

$$\frac{\partial \alpha}{\partial s} = \frac{\partial \alpha}{\partial \phi} = \frac{\psi - \psi_0}{(\phi - \phi_0)^2 + (\psi - \psi_0)^2}$$

when $\phi + i\psi$ is a point on a horizontal side, and

$$\frac{\partial \alpha}{\partial s} = \frac{\partial \alpha}{\partial \psi} = \frac{-(\phi - \phi_0)}{(\phi - \phi_0)^2 + (\psi - \psi_0)^2}$$

when $\phi + i\psi$ is a point on a vertical side.

Henceforth we shall write

$$U = U(\phi, \psi) = \ln q, \quad V = V(\phi, \psi) = -\theta.$$

Consider equation (17) (equation (10) of Lauck [9]) and equation (18) or (19) (equation (18) of Jaswon [7]). When $\rho$ is small, $-\ln \rho$ is much smaller than $1/\rho$. Hence we can expect to obtain greater accuracy using (19) instead of (17). However, we still need a particular formula for integration when the integral contains the singular point $w_0 = \phi_0 + i\psi_0$.

Using the Taylor series expansion, we have

$$\begin{aligned}
&\int_{-h}^{h} \frac{\partial U}{\partial \phi} \ln \phi \, d\phi \\
&= \int_{-h}^{h} \left[\left(\frac{\partial U}{\partial \phi}\right)_0 + \phi\left(\frac{\partial^2 U}{\partial \phi^2}\right)_0 + \frac{\phi^2}{2}\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 + \frac{\phi^3}{6}\left(\frac{\partial^4 U}{\partial \phi^4}\right)_0 + \dots\right] \ln|\phi| \, d\phi \\
&= 2h(\ln h - 1)\left(\frac{\partial U}{\partial \phi}\right)_0 + \frac{h^3}{9}(3 \ln h - 1)\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 + O[h^5(5 \ln h - 1)],
\end{aligned} \tag{24}$$

where the subscript 0 refers to the point $(\phi_0, \psi_0)$.

Suppose that $w_0 = \phi_0 + i\psi_0$ is a point near a corner, say $B = (\phi_0 + h_x) + i\psi_0$ in Fig. 1, on side $AB$, and we wish to integrate

$$\int_B^{B+h_y} U(\phi_0 + h_x, \psi) \frac{\partial \alpha}{\partial \psi} d\psi$$

in the second integral of (21). If $h_x$ and $h_y$ are small, it appears that $\partial\alpha/\partial\psi$ will increase rapidly as the point $\phi + i\psi$ approaches $B$ along $CB$ and attains its maximum value at $B$. Hence, we need a more accurate formula for that integral. Applying the Taylor expansion for $U(\phi_0 + h_x, \psi)$ at $B$, we obtain

$$\begin{aligned}
&\int_0^{h_y} U(\phi_0 + h_x, \psi) \frac{\partial \alpha}{\partial \psi} d\psi \\
&\quad = \int_0^{h_y} \left[ (U)_B + \frac{\psi^2}{2!}\left(\frac{\partial^2 U}{\partial \psi^2}\right)_B + \frac{\psi^4}{4!}\left(\frac{\partial^4 U}{\partial \psi^4}\right)_B + \frac{\psi^6}{6!}\left(\frac{\partial^6 U}{\partial \psi^6}\right)_B + \dots \right] \\
&\quad \times \frac{h_x}{h_x^2 + h_y^2} d\psi = \alpha_1 + \left[ 2\left(\frac{h_x}{h_y}\right) - \left(\frac{h_x}{h_y}\right)^2 \alpha_1 \right] \frac{h_y^2}{2!}\left(\frac{\partial^2 U}{\partial \psi^2}\right)_B \\
&\quad + \left[ \frac{8}{3}\left(\frac{h_x}{h_y}\right) - 2\left(\frac{h_x}{h_y}\right)^3 + \left(\frac{h_x}{h_y}\right)^4 \alpha_1 \right] \frac{h_y^4}{4!}\left(\frac{\partial^4 U}{\partial \psi^4}\right)_B \\
&\quad + \left[ \frac{32}{5}\left(\frac{h_x}{h_y}\right) - \frac{8}{3}\left(\frac{h_x}{h_y}\right)^3 + 2\left(\frac{h_x}{h_y}\right)^5 - \left(\frac{h_x}{h_y}\right)^6 \alpha_1 \right] \frac{h_y^6}{6!}\left(\frac{\partial^6 U}{\partial \psi^6}\right)_B + \dots
\end{aligned} \tag{25}$$

where $\tan \alpha_1 = 2h_y/h_x$ and $(\partial^{2n+1} U/\partial \psi^{2n+1})_B$, $n = 0, 1, 2, \dots$, are assumed to be zero.

As we are going to compute $\partial U/\partial s$, $\partial^2 U/\partial s^2$, $\partial^3 U/\partial s^3$ and so on, we need the values of $U$ and $V$ along $AB$, $BC$, $CD$ and $DA$, and also outside the rectangle $ABCD$ (See Fig. 2). Since $V = 0$ on $AB$, $BC$ and $DA$, we may apply the principle of reflection to obtain the necessary values of $U$ along $AA'$, $AA''$, $BB'$, $BB''$, $CC'$ and $DD'$, and $V$ along $CC'$ and $DD'$. For values of $U$ on $CC''$ and $DD''$, we apply the reflection principle across the free streamline (see Appendix). Using the Taylor series expansion, we have

$$\begin{aligned}
60h_x\left(\frac{\partial U}{\partial \phi}\right)_0 &= 45(U_1 - U_{-1}) - 9(U_2 - U_{-2}) + (U_3 - U_{-3}) + O(h^7), \\
180h_x^2\left(\frac{\partial^2 U}{\partial \phi^2}\right)_0 &= 270(U_1 + U_{-1}) - 27(U_2 + U_{-2}) + 2(U_3 + U_{-3}) \\
&\quad - 490U_0 + O(h^8), \qquad (26) \\
8h_x^3\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 &= -13(U_1 - U_{-1}) + 8(U_2 - U_{-2}) - (U_3 - U_{-3}) + O(h^7),
\end{aligned}$$

where $U_j = U(\phi_0 + jh_x, \psi_0)$, and again the subscript 0 refers to the point $(\phi_0, \psi_0)$.

The same formula can be used for $\partial U/\partial \psi$, $\partial V/\partial \phi$, $\partial V/\partial \psi$ etc. The derivatives $(\partial^{2n} U/\partial \psi^{2n})_B$, $n = 1, 2, 3$, in (25) can be evaluated using the following:

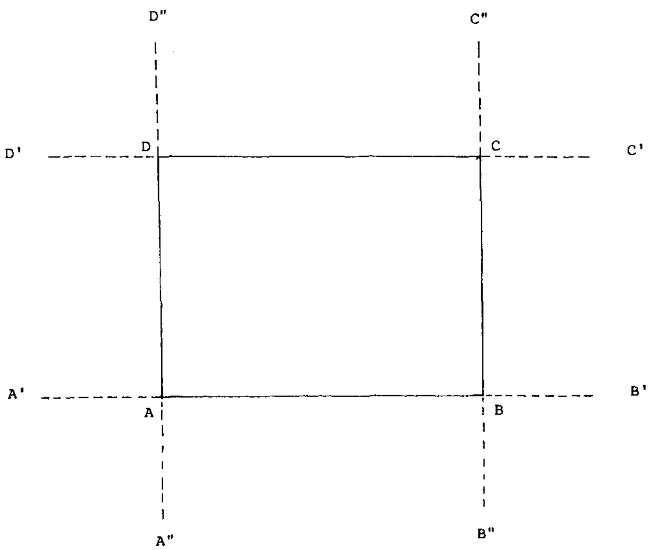


Figure 2. Reflections in the $w$-plane.

$$\begin{aligned}
180\,\frac{h_y^2}{2!}\left(\frac{\partial^2 U}{\partial\psi^2}\right)_B &= 270U_1 - 27U_2 + 2U_3 - 245U_0 + O(h_y^{14}),\\
72\,\frac{h_y^4}{4!}\left(\frac{\partial^4 U}{\partial\psi^4}\right)_B &= -39U_1 + 12U_2 - U_3 + 28U_0 + O(h_y^{14}),\\
360\,\frac{h_y^6}{6!}\left(\frac{\partial^6 U}{\partial\psi^6}\right)_B &= 15U_1 - 6U_2 + U_3 - 10U_0 + O(h_y^{14}).
\end{aligned} \tag{27}$$

We compute the integral in (4) by writing

$$\int_0^{nh} f(x)dx = \frac{h}{2}(f_0 + 2f_1 + \ldots + 2f_{n-1} + f_n) + \frac{h^2}{12}(f_0' - f_n') + O(h^4)$$

for small $n$ ($n = 2, 3, 4$) and

$$\begin{aligned}
\int_0^{nh} f(x)dx = {}& \frac{h}{2}(f_0 + 2f_1 + \ldots + 2f_{n-1} + f_n)\\
& - h[\tfrac{1}{12}(\nabla f_n - \Delta f_0) + \tfrac{1}{24}(\nabla^2 f_n + \Delta^2 f_0) + \tfrac{19}{720}(\nabla^3 f_n - \Delta^3 f_0)\\
& + \tfrac{3}{160}(\nabla^4 f_n + \Delta^4 f_0) + \tfrac{863}{60480}(\nabla^5 f_n - \Delta^5 f_0) + \tfrac{275}{24195}(\nabla^6 f_n + \Delta^6 f_0)] + O(h^8)
\end{aligned} \tag{28}$$

for large $n$ ($n > 4$), where $f_j = f(x = jh)$, see Fröberg [3].

Initial values of $U = \ln q$ and $V = -\theta$ along the sides of $ABCD$ were estimated, using smooth piecewise quadratic interpolation, remembering that $\theta = 0$ along $AB$, $BC$ and $AD$.

For the iterative method, the Gauss-Seidel procedure was used. New values of $U$ on $AB$, $BC$ and $DA$, and $V$ on $CD$, were obtained using (21), and $U$ at $A$ and $B$ (corners) using (26). The necessary derivatives of $U$ and $V$ were evaluated using (25). The free-surface condition then provided a new value of (dimensionless) $g$, since the current $V$ was known on $CD$, and $\phi = \phi_1$ and $q = q_1$ in (4). Knowing $g$, (4) was then used to find $U$ along $CD$ also, using (28), and the derivatives of $U$ on $CD$ by means of (25).

The procedure was repeated until successive approximations differed by a prescribed small number $10^{-k}$, where $k = 6$ in most cases, and $k = 4$ in a few cases. At a particular (boundary) point $(\phi, \psi)$, we computed $U$ or $V$ using (21) or (22). For convenience we define an "iteration" to be the computation of $U$ at all points on $DA$, $AB$ and $BC$, and $V$ at all points on $CD$; and a "cycle" to be three iterations together with evaluation of all necessary derivatives of $U$ or $V$ using (26), computing $U$ along $CD$ using the free surface condition (4), and computing the necessary derivates of $U$ using (26). In the next section, we shall discuss the number of cycles necessary for convergence.

## 5. Numerical results

It was mentioned, in Section 2, that we had normalized so that the flux $\psi_1 = 1$, and the speed in the trough $q_0 = 1$. We may then expect the iterative procedures outlined above to work well in the vicinity of $\phi_1 = 1$ and when $q_1$ is not too close to zero. When $\phi_1$ is large, shallow-water theory can be applied, and when $\phi_1$ is small, deep-water theory. When $q_1 = 1$, uniform flow is obtained, and as $q_1 \to 0$, the highest wave is approached.

It was found most convenient, for our purposes, to fix $\phi_1$ and $q_1$, and then compute $a$, $h$, $\lambda$, $c^2$, and $g$. For this reason, it was not easy to produce results that could be compared with

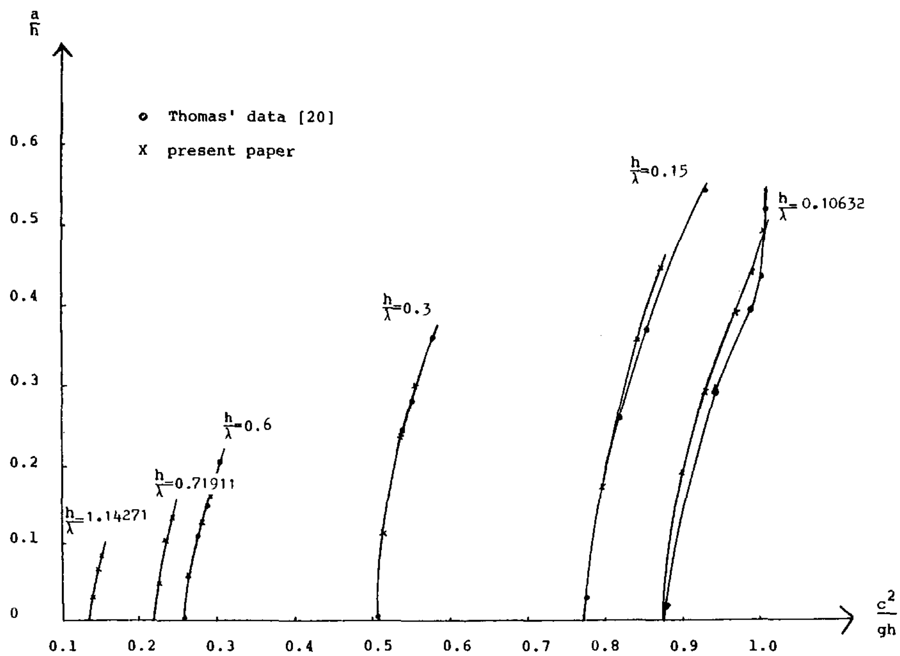


Figure 3.

TABLE 1

| $\frac{h}{\lambda}$ | $q_1$ | $\phi_1$ | $\frac{c^2}{gh}$ | $\frac{a}{h}$ | $\frac{a}{\lambda}$ |
|---|---|---|---|---|---|
| 1.14271 | 0.8 | 0.43791 | 0.14103 | 0.03058 | 0.03495 |
| | 0.6 | 0.43915 | 0.14725 | 0.06535 | 0.07568 |
| | 0.5 | 0.44 | 0.15221 | 0.08277 | 0.09458 |
| 0.71911 | 0.8 | 0.69623 | 0.22402 | 0.04858 | 0.03493 |
| | 0.6 | 0.69933 | 0.23386 | 0.10386 | 0.07467 |
| | 0.5 | 0.70150 | 0.24170 | 0.13157 | 0.09461 |
| 0.6 | 0.8 | 0.83465 | 0.26827 | 0.05817 | 0.03490 |
| | 0.6 | 0.83913 | 0.28007 | 0.12434 | 0.07460 |
| | 0.5 | 0.84223 | 0.28947 | 0.15754 | 0.09453 |
| 0.3 | 0.8 | 1.67173 | 0.51289 | 0.11109 | 0.03333 |
| | 0.6 | 1.68860 | 0.53551 | 0.23655 | 0.07096 |
| | 0.5 | 1.70019 | 0.55356 | 0.29895 | 0.08968 |
| 0.15 | 0.8 | 3.34841 | 0.79467 | 0.17030 | 0.02555 |
| | 0.6 | 3.39526 | 0.83938 | 0.35620 | 0.05343 |
| | 0.5 | 3.42430 | 0.87180 | 0.44512 | 0.06677 |
| 0.10632 | 0.8 | 4.72602 | 0.89777 | 0.18943 | 0.02014 |
| | 0.7 | 4.75467 | 0.92760 | 0.29086 | 0.03092 |
| | 0.6 | 4.79044 | 0.96671 | 0.39189 | 0.04167 |
| | 0.55 | 4.80856 | 0.98814 | 0.44067 | 0.04686 |
| | 0.5 | 4.82 | 1.00902 | 0.48965 | 0.05204 |

those of Thomas [20], which would require selection of the values of $\phi_1$ and $q_1$ so that resultant values of $h/\lambda$ agreed with those used by Thomas. However, this was done, if at the expense of extra computing time and the results are shown in Figure 3. It was observed, from the computed results, that $\phi_1$ is a *decreasing* function of $h/\lambda$, for *fixed* $q_1$. This made it easier to adjust $\phi_1$, when necessary, to produce a desired value of $h/\lambda$.

Two particular difficulties arose in the computing procedure adopted. The first was that the integrand of the second integral of (21) changed very rapidly for points near a corner, because of the form of $\partial\alpha/\partial\psi$. This difficulty was circumvented by using a formula (25) yielding greater accuracy. Secondly, the first integral of (21) became singular when the $j^{\text{th}}$ sub-interval contained 0. This, in turn, was overcome by expanding $\partial U/\partial\psi$ as Maclaurin series and integrating term-by-term, as shown in (24).

It was found that the lengths $h_x$ and $h_y$, the sub-intervals, had to be comparable, in order to obtain uniform accuracy around the contour. When considering shallow, or deep, water waves, it was found that it was not possible to obtain the same rapidity of convergence. However, we were principally interested in obtaining the profiles of waves in which the wave-length and depth were of the same order of magnitude.

The computed results are summarized in Tables 1, 2 and 3. Between three and five sets of

TABLE 2

| $\phi_1$ | $q_1$ | $a$ | $d$ | $h_1$ | $h$ | $g$ | $\lambda$ | $c$ | $c_1$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.43791 | 0.8 | 0.03359 | 0.01585 | 1.08263 | 1.09848 | 5.35830 | 0.96128 | 0.91110 | 0.91110 |
| 0.43915 | 0.6 | 0.07724 | 0.03367 | 1.14829 | 1.18197 | 4.14267 | 1.03436 | 0.84912 | 0.84912 |
| 0.44000 | 0.5 | 0.10023 | 0.04148 | 1.16944 | 1.21092 | 3.74158 | 1.05969 | 0.83043 | 0.83041 |
| 0.69623 | 0.8 | 0.05339 | 0.02520 | 1.07385 | 1.09906 | 3.37132 | 1.52836 | 0.91108 | 0.91108 |
| 0.69933 | 0.6 | 0.12302 | 0.05368 | 1.13106 | 1.18474 | 2.60130 | 1.64750 | 0.84896 | 0.84895 |
| 0.70150 | 0.5 | 0.15993 | 0.06634 | 1.14920 | 1.21554 | 2.34484 | 1.69034 | 0.83001 | 0.83000 |
| 0.83465 | 0.8 | 0.06395 | 0.03018 | 1.06915 | 1.09933 | 2.81461 | 1.83220 | 0.91109 | 0.91109 |
| 0.83913 | 0.6 | 0.14747 | 0.06433 | 1.12170 | 1.18603 | 2.17000 | 1.97673 | 0.84901 | 0.84901 |
| 0.84223 | 0.5 | 0.19182 | 0.07952 | 1.13804 | 1.21757 | 1.95500 | 2.02925 | 0.83009 | 0.83010 |
| 1.67173 | 0.8 | 0.12222 | 0.05722 | 1.04304 | 1.10026 | 1.47270 | 3.66759 | 0.91162 | 0.91185 |
| 1.68860 | 0.6 | 0.28157 | 0.12057 | 1.06973 | 1.19031 | 1.13650 | 3.96786 | 0.85114 | 0.85202 |
| 1.70019 | 0.5 | 0.36597 | 0.14811 | 1.07607 | 1.22418 | 1.02468 | 4.08062 | 0.83330 | 0.83455 |
| 3.34841 | 0.8 | 0.18667 | 0.08188 | 1.01422 | 1.09610 | 0.96429 | 7.30710 | 0.91648 | 0.91765 |
| 3.39526 | 0.6 | 0.41780 | 0.15603 | 1.01692 | 1.17294 | 0.76591 | 7.81978 | 0.86838 | 0.87235 |
| 3.42430 | 0.5 | 0.53356 | 0.18247 | 1.01621 | 1.19867 | 0.70283 | 7.99128 | 0.85701 | 0.86229 |
| 4.72602 | 0.8 | 0.20610 | 0.08239 | 1.00562 | 1.08801 | 0.87336 | 10.23364 | 0.92362 | 0.92525 |
| 4.75467 | 0.7 | 0.32608 | 0.11562 | 1.00546 | 1.12108 | 0.78202 | 10.54485 | 0.90180 | 0.90507 |
| 4.79044 | 0.6 | 0.44932 | 0.14174 | 1.00480 | 1.14653 | 0.71219 | 10.78367 | 0.88846 | 0.89337 |
| 4.80856 | 0.55 | 0.50958 | 0.15192 | 1.00445 | 1.15637 | 0.68439 | 10.87518 | 0.88432 | 0.88997 |
| 4.82000 | 0.5 | 0.57074 | 0.16270 | 1.00289 | 1.16559 | 0.65705 | 10.96620 | 0.87906 | 0.88668 |

values of $q_1$, $\phi_1$, $c^2/(gh)$ and $a/h$ have been found for each of five values of $h/\lambda$. These data have been plotted in Figure 3 and compared with those of Thomas. When $0.15 < h/\lambda < 0.6$, the results agree well, except for $h/\lambda = 0.15$ and small values of $q_1$. When $h/\lambda = 0.10632$, the two curves are quite different. When $h/\lambda > 0.6$, there are no corresponding values in Thomas' paper. No tables have been given by Schwartz [16], or Wehausen and Laitone [22], but a comparison with their graphs, when $0.0168 < h/\lambda < 0.3$, and $0.05 < h/\lambda < 0.6$, respectively, shows good agreement, producing smooth and comparable curves.

Table 3 shows the profile and surface inclination of the wave with $q_1 = 0.6$ and $\phi_1 = 4.79044$, corresponding to the third last line of Tables 1 and 2. It should be remembered that all quantities are dimensionless, so that the corresponding physical quantities are obtained by multiplying velocities by $q_0$ and lengths by $\psi_1/q_0$.

The iterative procedure was terminated when successive approximation agreed up to the sixth decimal place, which in most cases, required about 24 cycles. When $h/\lambda = 0.10632$, 36 cycles were required to obtain agreement up to the fourth decimal place.

## 6. Numerical checks

A number of checks were applied to the numerical procedure. First, the values of $y$ obtained by using (2) and (5) were compared, and in most cases were found to agree up to the seventh

TABLE 3

| $i$ | $q(i)$ | $\theta(i)$ (degrees) | $x(i)$ | $y(i)$ |
|---|---|---|---|---|
| 1 | 1.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 2 | 0.99953580 | 0.81871140 | 0.02997150 | 0.00021385 |
| 3 | 0.99814470 | 1.62781136 | 0.05996480 | 0.00085417 |
| 4 | 0.99582280 | 2.43966464 | 0.09000182 | 0.00192094 |
| 5 | 0.99255500 | 3.24859066 | 0.12010900 | 0.00341810 |
| 6 | 0.98833200 | 4.05569104 | 0.15030300 | 0.00534556 |
| 7 | 0.98313000 | 4.85686890 | 0.18061100 | 0.00770859 |
| 8 | 0.97692820 | 5.65442037 | 0.21105930 | 0.01050945 |
| 9 | 0.96969200 | 6.44248699 | 0.24167560 | 0.01375511 |
| 10 | 0.96139380 | 7.22404520 | 0.27249060 | 0.01744737 |
| 11 | 0.95198300 | 7.99092548 | 0.30353710 | 0.02159631 |
| 12 | 0.94142430 | 8.74661032 | 0.33485310 | 0.02620276 |
| 13 | 0.92964950 | 9.47965658 | 0.36647880 | 0.03127909 |
| 14 | 0.91661460 | 10.19342531 | 0.39846270 | 0.03682423 |
| 15 | 0.90223170 | 10.87192720 | 0.43085650 | 0.04285192 |
| 16 | 0.88644890 | 11.51694477 | 0.46372500 | 0.04935662 |
| 17 | 0.86915960 | 12.10580057 | 0.49713710 | 0.05635040 |
| 18 | 0.85031030 | 12.63601267 | 0.53118190 | 0.06381828 |
| 19 | 0.82978590 | 13.07435418 | 0.56595650 | 0.07176363 |
| 20 | 0.80755650 | 13.40623945 | 0.60158540 | 0.08015007 |
| 21 | 0.78354500 | 13.58242797 | 0.63820830 | 0.08895296 |
| 22 | 0.75783000 | 13.55874512 | 0.67600350 | 0.09808576 |
| 23 | 0.73053760 | 13.26347210 | 0.71516860 | 0.10744540 |
| 24 | 0.70211560 | 12.58638846 | 0.75594210 | 0.11682760 |
| 25 | 0.67342720 | 11.41934774 | 0.79856340 | 0.12592020 |
| 26 | 0.64600020 | 9.60817342 | 0.84323700 | 0.13425850 |
| 27 | 0.62246370 | 7.00052012 | 0.89002280 | 0.14113750 |
| 28 | 0.60619900 | 3.79540286 | 0.93864460 | 0.14574210 |
| 29 | 0.60 | 0.00000000 | 0.98836690 | 0.14746510 |

decimal place. Secondly, a number of trends were observed; assuming, as we did, that the streamlines crossing $AD$ are concave up, and those crossing $BC$ are concave down, then $q$ should decrease along the free surface from trough to crest, and around the contour $DABC$, attaining its maximum at $D$ and minimum at $C$. $\theta$ should rise to a single maximum between $D$ and $C$. Thirdly, and most importantly, the total momentum flux across $AD$ should be equal to that across $BC$. This result can easily be established using the following analogy of a lemma of Levi-Civita's [10, p. 276]:

$$\int_\Gamma q^2 dy = 2\int_\Gamma u(u dy - v dx) \tag{29}$$

where $q^2 = u^2 + v^2$, and $u$ and $v$ are conjugate harmonic functions. The lemma follows at once from Green's theorem.

When $\Gamma$ is the contour $ABCD$ in the $xy$-plane (Figure 1), we find that (in dimensional coordinates):

$$M_{AD} \stackrel{\text{def}}{=} \int_A^D (p+u^2)dy = \frac{1}{2}\left[\int_A^D u^2 dy + h_1(q_0^2 + gh_1)\right]$$

where $p + \frac{1}{2}(u^2 + v^2) + gy = \frac{1}{2}q_0^2 + gh_1 = \frac{1}{2}q_1^2 + gh_2$ and $h_2 = a + h_1$ is the depth $BC$.

Likewise

$$M_{BC} \stackrel{\text{def}}{=} \int_B^C (p+u^2)dy = \frac{1}{2}\left[\int_B^C u^2 dy + h_2(q_1^2 + gh_2)\right].$$

Applying (29) it is easy to show that $M_{AD} = M_{BC}$; that is, the total momentum flux across $AD$ is equal to that across $BC$.

Sample checks yielded the following comparisons (in dimensionless form):

| $q_1$ | $\phi_1$ | $M_{AD}$ | $M_{BC}$ |
|---|---|---|---|
| 0.6 | 0.83913 | 0.1002764 | 0.1002297 |
| 0.6 | 1.68860 | 0.1914649 | 0.1914360 |

## Appendix

The analytic continuation of harmonic functions across the free streamline is made possible by a method due to Lewy [24] based upon the following theorem which we quote in his notation:

Let $U(x, y)$ be harmonic near the origin in $y < 0$, and $V(x, y)$ a conjugate harmonic of $U$. Let $U$, $V$, $U_x$ exist and be continuous in the semi-neighbourhood of $y \leq 0$ of the origin. If the boundary values on $y = 0$ satisfy a relation of the form

$$U_y = A(x, U, V, U_x) \tag{B}$$

in which $A$ is an analytic function of all four arguments for all values occurring, then $U(x, y)$ and $V(x, y)$ are analytically extensible across $y = 0$.

We shall be concerned only with extension along the lines corresponding to $x$ = constant (actually $\phi$ = constant in our notation). It suffices then to say that the analytic extension is given by [24, equation $E_y$]

$$\frac{dG(0,y)}{dy} = \frac{dF(0,y)}{dy} - A\Big[iy, G(0,y) + F(0,y), i(G(0,y) - F(0,y)),$$
$$- i\frac{d}{dy}(G(0,y) + F(0,y))\Big] \tag{$E_y$}$$

where

$$2F(z) = U(x, y) + iV(x, y),$$

$$2G(0, 0) = U(0, 0) - iV(0, 0).$$

The analytic continuation of $F(z)$ along $x = 0$ is then given by

$$F(0, y) = \bar{G}(0, -y), \qquad y \geq 0,$$

where $G(0, y)$ is the solution of $(E_y)$ subject to the initial condition $G(0, 0) = U(0, 0) - iV(0, 0)$.

Changing to the notation of this paper, we replace $x$ and $y$ by $\phi$ and $\psi$ and $U$ and $V$ by $x$ and $y$ respectively. The free-surface condition (2) can be written in the form $(B)$ as

$$x_\psi = \left[\frac{1}{2g(h-y)} - x_\phi^2\right]^{\frac{1}{2}} = A[\phi, x, y, x_\phi] \qquad (B_\psi)$$

where $\phi$ and $x$ do not occur explicitly.

Then $(E_y)$ assumes the form (along $\phi = 0$):

$$\frac{dG}{d\psi} = \frac{dF}{d\psi} - A\left[\phi, x, i(G-F), -i\frac{d}{d\psi}(G+F)\right]$$

which simplifies to

$$\frac{dG}{d\psi} = \frac{1}{\dfrac{dF}{d\psi} 8g[i(G-F)-h]} \qquad (E_\psi)$$

where now

$$F(0, \psi) = x(0, \psi) + iy(0, \psi),$$

$$F(0, \psi) = \bar{G}(0, -\psi) \qquad \psi \geq 0.$$

Equation $(E_\psi)$ was integrated numerically along $\phi = 0$ and $\phi = \phi_1$, providing continuations of $x$ and $y$ across the free streamlines. It was found that the difference between the continuations obtained using Lewy's method and those obtained using Schiffman's method [15] (i.e. $g = 0$) was quite small, and did not materially effect the shape of the wave except for an occasional change in the sixth decimal place.

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